Worksheet 5A Solutions

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1. Vectors in Cartesian Coordinates

   (a) How do you write an arbitrary vector $\vec{A}$: $length = |\vec{A}|$
   Take the square root of the sum of the squares of its individual components or: $|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$

   (b) Draw vectors, $\vec{B} = 2\hat{i} - \hat{j} + 3\hat{k}$, and $\vec{C} = -\hat{i} + 2\hat{j} - \hat{k}$.
   Label the positive directions for the x,y, and z axes. To draw any vector, move in the x direction the number of ticks in front of $\hat{i}$, in the y direction move the number of ticks in front of $\hat{j}$ and in the z direction, move the number of ticks in front of $\hat{k}$.

   (c) What is $\vec{B} - \vec{C}$?
   $\vec{B} - \vec{C} = (2\hat{i} - \hat{j} + 3\hat{k}) - (-\hat{i} + 2\hat{j} - \hat{k}) = 3\hat{i} - 3\hat{j} + 4\hat{k}$ **The length of the vector** is $\sqrt{3^2 + (-3)^2 + 4^2}$

2. Vectors in Spherical Coordinates

   (a) What is x in terms of r, θ, and ϕ?
   $x = r\sin(\theta)\cos(\phi)$

   (b) What is y in terms of r, θ, and ϕ?
   $y = r\sin(\theta)\sin(\phi)$

   (c) What is z in terms of r, θ, and ϕ?
   $z = r\cos(\theta)$

3. Volume Elements

   (a) What is the expression for the volume element used in the 3D integrals: $dV = dxdydz$?
   $dV = r^2\sin(\theta)drd\theta d\phi$

   (b) Does your volume element yield the correct value for the volume of a sphere if you perform the integral, $V = \int dV$? What is the explicit form of this integral in terms of r, θ, and ϕ?
   $\int dV = \int_0^R \int_0^\pi \int_0^{2\pi} r^2\sin(\theta)drd\theta d\phi = \frac{4}{3}\pi R^3$

   (c) Why is $drd\theta d\phi$ alone not the appropriate volume element?
   A volume element must have units of cubic length whereas the above has units of only length. Angles must be paired with a distance from a center to form a length.

   (d) What is the expression used to evaluate the average value of $\cos(\theta)$ over the surface of sphere?
   A differential area is given by $dA = r^2\sin(\theta)d\theta d\phi$. We must weigh $\cos(\theta)$ by whatever fraction of total area the differential area occupies:

$$< cos(\theta) >= \frac{\int_0^\pi \int_0^{2\pi} \cos(\theta)r^2\sin(\theta)d\theta d\phi}{\int_0^\pi \int_0^{2\pi} r^2\sin(\theta)d\theta d\phi} = \frac{\int_0^\pi \int_0^{2\pi} \cos(\theta)r^2\sin(\theta)d\theta d\phi}{4\pi r^2} \quad (1)$$

(e) What is the expression used to evaluate the average of $\cos^2(\theta)$ over the surface of a sphere?
Using the same reasoning:

$$< \cos^2(\theta) >= \frac{\int_0^{\pi} \int_0^{2\pi} \cos^2(\theta) r^2 \sin(\theta) d\theta d\phi}{4\pi r^2} \tag{2}$$