BRENT EDWARDS

Have You Lost Your Marbles?

Three Creative Problem-Solving Approaches

As an eighth-grade mathematics teacher, I have come to value the process of communicating about mathematics. I have learned that it is vital for students to explain and describe their work, not just simply submit their solutions. I use several methods to encourage my students to share their problem-solving processes, such as asking them to show their work on the board for the class, write essays, and answer a simple question like "How did you arrive at your answer?" Of these tasks, writing takes the most time and effort, but the results can be extraordinary. The formal essay requires students to organize their thoughts, prepare a logical sequence, and explicitly communicate the details of their work. This "thinking about thinking" translates to an improved problem-solving ability and greater conceptual understanding.

I use a Problem of the Week (POTW) as a way to prompt reflective thought and clear communication (see **fig. 1**). To encourage deliberation and careful work, I assign the POTW on a Friday, make it due the following Friday, and assign it a relatively high point value. That is, it is worth more then a standard homework assignment, in terms of points possible, and is worth roughly half of a test grade. Although the correct solution is important, 70 percent of the available points come from a clear and detailed explanation of the student's solution process, including any failed or incorrect methods, and a justification for the chosen solution. This approach can be challenging to students; many arrive at the correct solution but do not provide a meaningful explanation and are surprised at the poor grade that results. Other students arrive at an incorrect solution but explain

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This department shares the thinking of middle school students as they explore and communicate mathematics. It highlights students' work, including projects, investigations, or creative solutions to the problems in the "Food for Thought from Jay's Diner" (currently called "Solve It!") and "Menu of Problems" departments. Original student work is included, along with sufficient information about the activity so that readers can try the ideas with their students. Please send manuscripts to MTMS, *NCTM, 1906 Association Drive, Reston, VA 20191-1502.*

Problem of the Week Guidelines

This is supposed to be a harder problem that takes some real thought. Don't wait until the night before it's due to start work!

The POTW is worth 10 points. Here's the point breakdown:

- **State** the problem correctly. You may copy it word for word or put it into your own words. Just make sure it is an accurate version of the problem. *1 point*
- **Answer** the problem correctly. I expect this to be in one or two complete sentences. Even if you aren't sure of the answer, give your best guess. *2 points*
- **Explain** the process you used to solve the problem. If you used scrap paper, attach it, but be sure to write out and explain what you did. If you tried something and it didn't seem to work, explain that, too! These problems will get hard enough that you won't solve them all on the first go. This should be *at least* a paragraph, and probably two or three. *5 points*
- **Justify** your answer. Explain why you believe it is correct, or why you chose that answer instead of a different possibility. Is your answer reasonable? Does it answer the question? Are there any other possible answers? *2 points*

I expect that a well-written POTW will be one full page. You don't have to use a word processor but it does help make your paper look nicer. Attach any diagrams, models, or scrap paper along with your written pages—show me how you figured it out.

Notice that getting the answer right is worth only 2 out of 10 points. If you get a wrong answer but really explain it well and provide a good reason for why you believe it is correct, you could still get 8/10. If you get the right answer but don't explain or justify it, you could get just 2/10. Explain! I am more interested in how you think, how you figure things out, and how you explain yourself than just whether you got it right.

You may talk about this with your family members, but I want *you* to do the work of figuring this out. Please don't talk about the problem with other students, no matter what.

I will grade this according to the points listed above, and I will also occasionally grade these on the 6 Traits of Writing. Take your time, think, and write well.

Fig. 1 POTW guidelines

their thought processes thoroughly, provide a rational justification, and earn a majority of the available points. This approach allows students to see the value of both process and product when solving problems.

For a recent POTW, I used the following problem that appeared in the September 2004 issue of this journal:

> Andy was on his way to a marble tournament when he met Bob. Andy's bag of marbles was heavy, so he gave Bob 1/2 of his marbles plus 2. Then he met Alex and gave him 1/4 of his remaining marbles plus 5. Just before arriving at the tournament, he met Emily and gave her 1/2 of the marbles he had left. When he got to the tournament, he had 2 marbles left. How many marbles did he start with?

I assigned the problem to approximately 130 students who were in my five classes.

Guess and Check

MOST STUDENTS USED A GUESS-AND-CHECK method and approximately half the students found the correct answer. Many students realized that the original amount of marbles had to be an even number, because a fractional result was unreasonable in this context. A common strategy was to guess an amount, say, 20, work forward through the steps outlined in the problem, and then revise the guess. Several students realized that if their outcome was less than 2, then their original amount needed to be higher. If they finished with a result greater than 2, then their guess was revised downward. They continued this process until they narrowed the search down and found that 28 was the correct solution. Although this is a valid solution strategy, it is not elegant and is potentially time-consuming. For example, if a fraction had been reasonable, the guess-and-check method could have taken an unreasonably long time.

Work Backward

THREE STUDENTS DISPLAYED CLEVER AND ELEgant strategies, working backward to arrive at the solution. They all used equations to model the situation, but the specifics of each approach varied in interesting ways. Their approaches follow.

Kelsey's strategy

Kelsey Samuelsen used three equations, one equation to represent each of Andy's transactions (see **fig. 2**). Rather than focus on how many marbles Andy lost at each stage, she tracked how many marbles he *retained* during each encounter. She wrote that if Andy began with x marbles, "He gave Bob 1/2 of x, + 2, so he had $1/2x - 2$." She assigned this value to y. In his next encounter, she wrote, "Andy gave Alex 1/4 of y + 5, so he has $3/4y - 5$," which she assigned to z. In his final transaction, he "gave Emily 1/2 of z," which left him with 2. In doing so, she essentially created the following system of equations, with x representing the initial amounts of marbles:

$$y=\frac{1}{2}x-2$$

$$z=\frac{3}{4}y-5$$

$$\frac{1}{2}z=2$$

Andy has x marbles
He gave Bob $\frac{1}{2}$ of x, + 2 = he has $\frac{1}{2}x - 2$ marbles = y
He gave Alex $\frac{1}{4}$ of y + 5 = he has $\frac{3}{4}y - 5$ = z
He gave Emily $\frac{1}{2}$ of z
He had 2 marbles left

Go BACKWARD
He had 2, which is $\frac{1}{2}$ of before Emily
So $2 \times 2 = 4$
Since $4 = \frac{3}{4}y - 5$ you add 5 to both sides
$+5 \quad +5$
$9 = \frac{3}{4}y$
$\frac{3}{4} \quad \frac{3}{4}$
$12 = y$
12 marbles is what Andy had before Alex.
Lastly, $12 = \frac{1}{2}x - 2$ so you solve
$+2 \quad +2$
$14 = \frac{1}{2}x$
$\frac{1}{2} \quad \frac{1}{2}$
$28 = x$
28 is the answer

$2 \times 2 = 4$
$4 = \frac{3}{4}y - 5$
$+5 \quad +5$
$9 = \frac{3}{4}y$
$\frac{3}{4} \quad \frac{3}{4}.$
$12 = y$
$12 = \frac{1}{2}x - 2$
$+2 \quad +2$
$14 = \frac{1}{2}x$
$\frac{1}{2} \quad \frac{1}{2}$
$28 = x$

Fig. 2 Kelsey Samuelsen's strategy

She then solved the equations by substitution. After solving the third equation for $z = 4$, she substituted 4 for z in the second equation. She then solved it for $y = 12$. Substituting that value into the first equation, she found $x = 28$ as the initial amount of marbles. For her to establish and solve this system intuitively was quite impressive, since we had not yet studied systems of linear equations.

Kristofer's strategy

Kristofer Christakos began by trying to express the entire process with a single equation. He created the equation $(x \bullet 1/2 - 2)(1/4 - 5)1/2 = 2$ to model the series of transactions, with x again representing the initial amount of marbles. When solved manually, this equation gave him an answer of 78, which he then checked by following the transaction listed in the story. Starting with 78 gave him 11.375 marbles at the end, which he concluded "was obviously not 2 marbles." He tried entering his equation as written into a calculator and received an answer of –87 7/8, again, obviously incorrect. He wrote, "That's when I got the idea of combining the working backward strategy with the equation strategy."

He began with Emily's transaction, deriving the equation $C \bullet 0.5 = 2$, with C representing "the number of marbles in the bag at the current point of time in the story, working backward." Solving this equation for C, he got 4 marbles "before [Andy] met Emily and gave her half." Moving forward from Emily, Kristofer did the best job of explicitly focusing on the marbles remaining in Andy's possession. He wrote the following explanation:

> I wanted to find out how many marbles Andy had, not how many he has given away. If Andy gave 1/4, then he should have 3/4. Then Andy gave 5, so 3/4 of the current marbles at this point working backward (B), subtracted by the other 5 marbles, should equal the number of marbles he has after this (4).

Translating this information into an equation, he developed $3/4B - 5 = 4$ and solved for B, which is 12. He continued, "So the amount of marbles Andy had before he gave them away to Alex is 12." In his transaction with Bob, Andy gave away half his marbles plus 2. Kristofer then explained:

> Once again, I wanted to find how many marbles Andy had at the point in the story, not how many he gave away. So if Bob got 1/2, then Andy still had the other half. Then, Andy gave Bob 2, so Andy had 1/2 marbles (A) at this point in the study, subtracted by the 2 he also gave Bob. That means $1/2A - 2 = 12$, which solved to $A = 28$.

In checking his solution, Kristofer made a graphical representation of both his process working backward from 2 marbles and of Andy's transactions working down from 28 marbles (see **fig. 3**). Both checked the work, proving his solution.

Both Kristofer and Kelsey deliberately chose to model each transaction independently and use the results from one equation as a value in the next to ultimately arrive at the final solution.

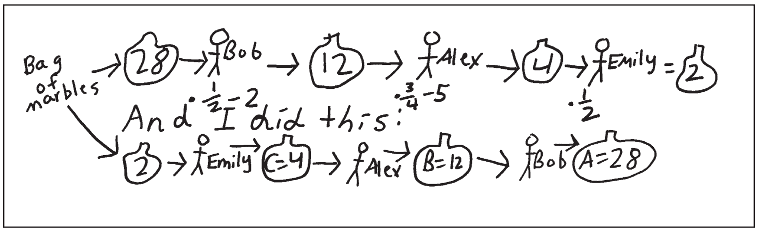

Fig. 3 Kristofer Christakos's strategy

Andrew's strategy

In contrast, Andrew Roche chose to represent the entire series of transactions in one equation. Initially, he solved the problem by guess and check but then realized he could derive a single equation to model the situation. He went through several false steps, including distributing 1/4 before realizing that 3/4 correctly represented the amount of marbles that Andy had remaining (see **fig. 4**). He also went through some errors with the order of operations before arriving at the correct equation:

$$2=\left[\left(\frac{x}{2}-2\right)\frac{3}{4}-5\right]\cdot\frac{1}{2}$$

Of all my students, Andrew was the only one to correctly link all elements of this problem into one coherent, accurate equation.

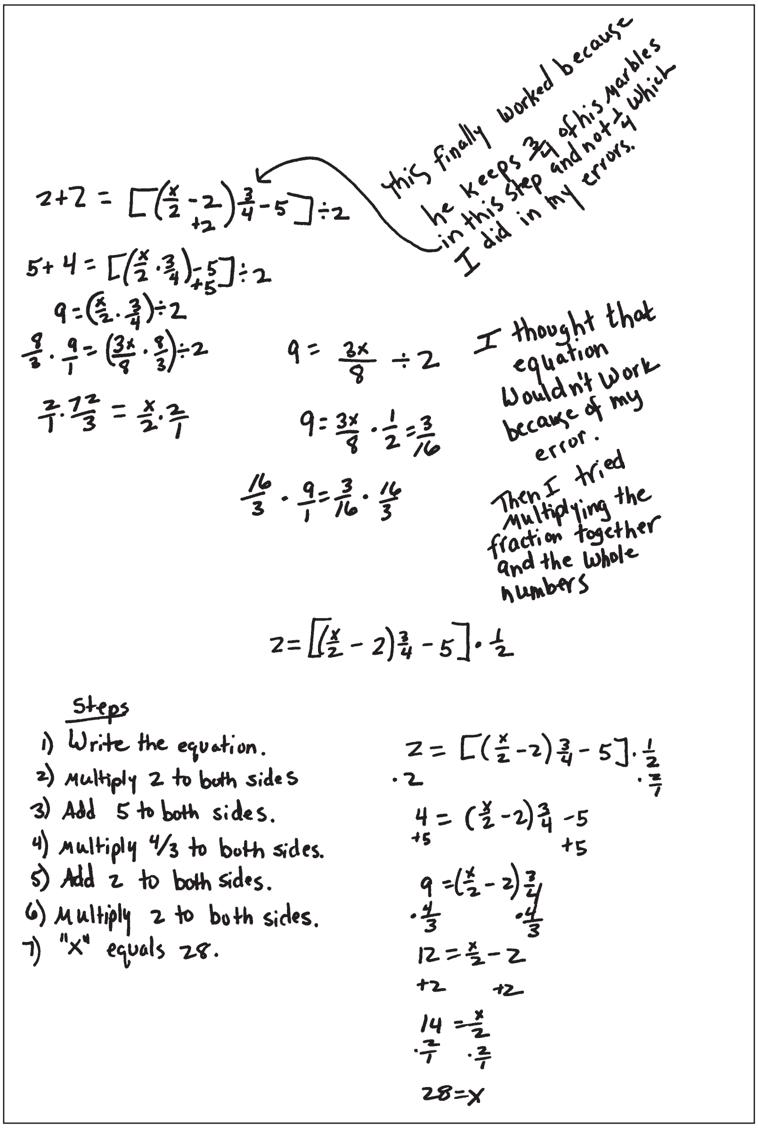

Fig. 4 Andrew Roche's strategy

Closing Comments

OF THE STUDENTS WHO FOUND AN INCORRECT solution, many arrived at either 44 marbles or 78 marbles for Andy's initial amount. Both are reasonable answers to the original question, but neither checks out when the story is followed. It was enlightening (and discouraging) to me that so many of my students failed to check their answers. This situation clearly presents an opportunity to improve my instruction and stress the importance of checking solutions. There were also a few students who were not troubled by odd answers or even negative answers, both of which are clearly impossible in the context of this problem. I will work on improving students' number sense and their ability to evaluate the reasonableness in their answers.

Overall, I was pleased with the depth of thought produced by this problem. It generated three popular answers—28, 44, and 78—and several interesting solution methods. I was also pleased with the quality of the writing that was generated and the overall caliber of the students' work. It is vital for students to fully grasp the importance of the solution process, the value of checking solutions, and the reasonableness of the solution itself. We will continue to employ a Problem of the Week twice monthly, and I encourage other educators to use extension problems such as this one in their classroom. I also encourage *MTMS* to continue to publish such interesting challenges. □